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# Homogeneous intrusions in a rotating stratified fluid

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A study is made of intrusions of fixed volumes of fluid of constant density into a uniformly stratified and uniformly rotating environment. In particular, the eventual steady-state configuration is sought for the ideal fluid case. Exact two-dimensional solutions can be found using a co-ordinate transformation which converts the equations satisfied outside the intrusion into the Cauchy–Riemann equations. The same technique does not, however, yield linear equations in the axisymmetric case.

Effects of friction are also considered for cases where the quasi-geostrophic approximation can be made. Vertical advection by the viscosity-induced motion is shown to have the same effect on the temperature field as a greatly enhanced lateral diffusion, and therefore tends to spread the intrusion out laterally.

## 1. Introduction

Intrusive features are found in the ocean where contrasts in water properties occur, i.e. in the neighbourhood of fronts, etc. For instance, Gregg (1976) has discussed such features near the surface, and intrusions found near fronts have been discussed by Horne (1978). McCartney, Worthington & Raymer (1980) have found isolated blobs of Labrador Sea Water many thousands of kilometres from their source regions, showing that such features can have a long lifetime. Armi (1978) and Armi & d'Asaro (1980) have found relatively homogeneous lenses of fluid which appear to have been formed by eruption of the bottom mixed layer into the interior. Such lenses probably play an important part in the way mixing takes place in the ocean, so there is some interest in finding techniques for examining their behaviour.

Consider the following situation as an idealized formulation of the problem. A stratified fluid with uniform buoyancy frequency $N$ is at rest relative to a frame rotating with constant angular velocity $\frac{1}{2}f$ about a vertical axis and is assumed to be of infinite extent. Into this environment, a finite volume of fluid with constant temperature (which can be taken as zero by making it the reference value) is intruded. The problem is to calculate the equilibrium configuration of the intruded fluid, the distortion of the temperature field of the surrounding fluid, and the associated velocity field, assuming that friction and mixing effects can be ignored.

It is the above idealized problem which will be dealt with in this paper or, rather, a two-dimensional version of it, followed by some discussion of effects of friction, the axisymmetric case, etc. The ideal problem is closely related to the one considered by Rossby (1938) which clearly demonstrated how a rotating fluid adjusts to equilibrium under gravity. A review of the subject has been given by Blumen (1972), and Charney (1973) discusses this fundamental problem in the book *Dynamical Meteorology*.

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Rossby (1938) considered an initial-value problem where the fluid at some given time has a non-equilibrium configuration, and then is allowed to adjust under gravitational forces. This may be regarded as equivalent to an extreme case of the idealized intrusion problem in which all the new fluid is inserted very rapidly in a finite time. At the end of this time, the fluid would have a non-equilibrium configuration so would then adjust to equilibrium by the same processes as occur in Rossby's classical problem. The significant feature of this adjustment is that the final equilibrium state does not depend on details of transient behaviour, but only on the distribution of potential vorticity in the initial state. The conclusion is that, in order to calculate the solution of the intrusion problem formulated above, it is necessary to know the potential vorticity distribution in the intruded fluid at some initial time. If this is known, experience with the Rossby problem would suggest that the equilibrium state would then be uniquely determined.

In contrast to the Rossby problem which corresponds to a rapid insertion of new fluid, the intrusion could be made so slowly that the fluid is, at every stage of the process, very close to an equilibrium state. An example of this limit was studied by Gill *et al.* (1979) where a fluid of one fixed density was slowly intruded into a fluid of another density. In this case, the configuration which obtains at any given time depends on the angular momentum of the fluid which has been added.

The approach adopted below to finding the equilibrium is somewhat pragmatic because of the difficulties inherent in solving a nonlinear problem of the type specified above, so attention is concentrated on the configuration of the fluid *outside* the intrusion rather than in the intrusion itself. The value of Ertel's potential vorticity (Eliassen & Kleinschmidt 1957) in this fluid retains its original uniform value which means that, following Hoskins & Bretherton (1972), a co-ordinate transformation can be found which reduces the equations to Laplace's equation in the outside fluid, so making the problem linear in transformed space. The condition to be applied on the boundary of the intruded lens turns out to be exactly the same as the one for potential flow past an obstacle, so it is possible to draw on the set of available solutions for this problem. The potential flow solutions, however, are for given *shapes* of intruded lenses in a transformed co-ordinate space rather than for given distributions of potential vorticity or angular momentum. Nevertheless, the classical solutions for potential flow past *elliptical* obstacles are taken and reinterpreted as solutions of an intrusion problem. It turns out that they correspond to intrusions which have uniform vorticity and include a case where the intrusion has zero angular momentum and zero potential vorticity. Such an intrusion could be produced in the laboratory by introducing fluid through a narrow orifice with near-zero angular momentum.

The idealized two-dimensional intrusion problem is studied in §§2–7. First the equations are derived, and the transformation which reduces them to the Cauchy–Riemann equations. This is followed by a discussion of the solutions for various shapes of boundaries. Later sections give a brief discussion of some effects which would be expected to occur in real fluids, and of the axisymmetric problem.

## 2. Equations

Before any new fluid is intruded, it is assumed that the whole domain is filled with an incompressible fluid which is at rest relative to a frame of reference rotating with uniform angular velocity $\frac{1}{2}f$ about a vertical axis. This fluid is assumed to be uniformly stratified, that is, to have uniform buoyancy frequency $N$ and hence temperature $\theta$ given by

$$\alpha g \frac{d\theta}{dz} = N^2, \tag{2.1}$$

where $z$ is the vertical co-ordinate, $\alpha$ is the thermal expansion coefficient (assumed constant) and $g$ the acceleration due to gravity.

Then, over some unspecified period of time, new fluid with a fixed uniform temperature is inserted. For convenience, the reference temperature is taken so this temperature is zero and the origin of the vertical co-ordinate $z$ is chosen such that the equilibrium centre of mass of the intruded fluid is at $z = 0$. Details of the insertion process are not known, nor are details of the adjustment to equilibrium. Instead, the problem considered here is to find possible equilibrium solutions for such a system.

For simplicity, attention is restricted initially to two-dimensional problems, that is, where the fluid properties depend only on two co-ordinates $(x, z)$ where the $x$ axis is horizontal and fixed in the rotating reference frame. Consider now the equations which must be satisfied in the equilibrium state. First, there is the hydrostatic equation

$$\frac{\partial p}{\partial z} = \alpha g \theta, \tag{2.2}$$

where $p$ is (1/density) times the pressure perturbation from that which a fluid of uniform density and zero temperature would have. Secondly, the fluid would settle down to a geostrophic equilibrium where the velocity $v$ normal to the plane $(x, z)$ is related to the horizontal pressure gradient by the equation

$$fv = \frac{\partial p}{\partial x}. \tag{2.3}$$

If pressure is eliminated from these two equations, they may be replaced by a single equation, namely the thermal-wind equation

$$f \frac{\partial v}{\partial z} = \alpha g \frac{\partial \theta}{\partial x}. \tag{2.4}$$

There is one further equation which must be satisfied in the fluid outside the intruded lens, and this comes from the property that Ertel's potential vorticity $q$ (see, for example, Eliassen & Kleinschmidt 1957) is conserved during any adjustment process of an ideal fluid. The quantity $q$ is given by the scalar product of the absolute vorticity and the temperature gradient, that is by

$$q = \left(f + \frac{\partial v}{\partial x}\right) \frac{\partial \theta}{\partial z} - \frac{\partial v}{\partial z} \frac{\partial \theta}{\partial x} \tag{2.5}$$

and, for each material particle, must have the same value as it did before the intrusion took place. In other words, $q$ must have the *uniform* value

$$q = \frac{fN^2}{\alpha g}, \tag{2.6}$$

everywhere outside the intruded lens. The fact that $q$ is constant greatly simplifies the problem.

The equilibrium solution for $(v, \theta)$ which is sought must therefore satisfy two equations, namely the thermal-wind equation (2.4) and the potential vorticity equation (2.5), with $q$ having the uniform value (2.6). The boundary conditions to be satisfied are that the perturbation produced by the intrusion should decay to zero at large distances from the lens, and that $\theta$ should be zero on the lens boundary. This is a nonlinear problem, but can be transformed into a linear one by a change of co-ordinates, as will be shown in the next section.

## 3. Co-ordinate transformation

Hoskins & Bretherton (1972) have shown how the above equations can be reduced to Laplace's equation by using the 'semi-geostrophic' co-ordinates first introduced by Yudin (1955). Before discussing this transformation, it is instructive to write the two relevant equations (2.4) and (2.5) with the variable $v$ replaced by the variable $M$ defined by

$$M = fx + v. \tag{3.1}$$

Changes in $M$ are equal to changes in angular momentum about a distant axis divided by distance from that axis, and so $M$ is an analogue for two-dimensional flows of angular momentum. With this definition, it is obvious that (2.5) takes the Jacobian form

$$\frac{\partial(M, \theta)}{\partial(x, z)} = q. \tag{3.2}$$

It is perhaps not so obvious that the thermal wind equation can also be written in Jacobian form, namely

$$\left.\begin{aligned} f\frac{\partial(M, x)}{\partial(x, z)} &= -\alpha g \frac{\partial(\theta, z)}{\partial(x, z)}, \\ \text{or} \qquad f\frac{\partial(M, x)}{\partial(\theta, z)} &= -\alpha g. \end{aligned}\right\} \tag{3.3}$$

The conversion of the thermal wind equation (2.4) into the Jacobian form (3.3) shows how the two Jacobians (3.2) and (3.3) can be transformed into the Cauchy–Riemann equations: for if one variable (in this case $z$) from the top line of the Jacobian and one variable (in this case $x$) from the bottom line are used as independent variables, the Jacobian takes the linear form of the original thermal wind equation (2.4). A simultaneous reduction of both Jacobians to linear form occurs using either $(M, z)$ or $(x, \theta)$ as independent variables for in each case one variable occurs in the top line and one in the bottom line of *both* Jacobians. This explains the 'duality' between semi-geostrophic co-ordinates $(M, z)$ and isentropic co-ordinates $(x, \theta)$ noted by Hoskins & Draghici (1977).

In the present case, it is advantageous to retain $\theta$ as a *dependent* variable because the boundary condition on the edge of the lens is $\theta = 0$. Therefore semi-geostrophic co-ordinates $(M, Z)$ will be used as *independent* variables. Capital $Z$ will be used in this case to denote that a $Z$-derivative is taken with $M$ constant. Now (3.2) becomes

$$\left.\begin{aligned} \frac{\partial(M, \theta)}{\partial(M, Z)} &= q\frac{\partial(x, Z)}{\partial(M, Z)}, \\ \text{or} \qquad \frac{\partial \theta}{\partial Z} &= q\frac{\partial x}{\partial M} \end{aligned}\right\} \tag{3.4}$$

and, similarly, the thermal wind equation (3.3) becomes

$$\alpha g \frac{\partial \theta}{\partial M} = -f \frac{\partial x}{\partial Z}. \tag{3.5}$$

It simplifies the algebra if non-dimensional co-ordinates are now introduced. A vertical scale $H$ is chosen as the maximum elevation of the lens above its centre of gravity at $z = 0$. Then $NH/f$ is used as horizontal scale, $NH$ as the scale for velocity and also for the quantity $M$, $N^2H/\alpha g$ is used as temperature scale, $fN^2/\alpha g$ as potential vorticity scale and $N^2H^2$ is used as the scale for the variable $p$. These choices of scale have the effect of replacing all the constants in the equations by unity, and in particular (3.4) and (3.5) take the Cauchy–Riemann form

$$\frac{\partial \theta}{\partial M} = -\frac{\partial x}{\partial Z}, \quad \frac{\partial \theta}{\partial Z} = \frac{\partial x}{\partial M}. \tag{3.6}$$

If $x$ is eliminated, Laplace's equation results, that is

$$\frac{\partial^2 \theta}{\partial M^2} + \frac{\partial^2 \theta}{\partial Z^2} = 0. \tag{3.7}$$

The boundary conditions are that

$$\theta = 0 \tag{3.8}$$

on the lens boundary, and that at large distances from the lens

$$\frac{\partial \theta}{\partial Z} \to 1, \tag{3.9}$$

that is, the disturbance dies away and the temperature gradient tends to its undisturbed value.

The above equations and boundary conditions are sufficient to deal with the problem, but it is useful to note some further relationships which can be used if the pressure field needs to be calculated. For this purpose, the derivatives of $p$ with respect to the new independent variables are found as follows. First, the transformation from $(x, z)$ co-ordinates to $(M, Z)$ co-ordinates gives

$$\frac{\partial p}{\partial M} = \frac{\partial p}{\partial x}\frac{\partial x}{\partial M}, \quad \frac{\partial p}{\partial Z} = \frac{\partial p}{\partial x}\frac{\partial x}{\partial Z} + \frac{\partial p}{\partial z}. \tag{3.10}$$

Now the non-dimensional forms of (2.2) and (2.3) can be used to substitute $\theta$ for $\partial p/\partial z$ and $v$ for $\partial p/\partial x$ and the non-dimensional forms of (3.1), namely

$$x = M - v, \tag{3.11}$$

can be used to substitute for $x$. The result is

$$\frac{\partial p}{\partial M} = v - v\frac{\partial v}{\partial M}, \quad \frac{\partial p}{\partial Z} = \theta - v\frac{\partial v}{\partial Z}. \tag{3.12}$$

Hence, defining (Hoskins 1975)

$$\phi = p + \tfrac{1}{2}V^2, \tag{3.13}$$

it follows that

$$\frac{\partial \phi}{\partial M} = v, \quad \frac{\partial \phi}{\partial Z} = \theta, \tag{3.14}$$

and these can be regarded as the transformed version of the hydrostatic and geostrophic balance equations.

## 4. Solutions for a lens which is circular in transformed space

The problem to be solved for the fluid outside the lens has now become identical to that for irrotational flow past an obstacle, $\theta$ taking the place of stream function and $M, Z$ being the independent co-ordinates. Thus the complex quantity

$$w = x + i\theta, \tag{4.1}$$

is an analytic function of the complex variable

$$Y = M + iZ, \tag{4.2}$$

that is
$$w = w(Y). \tag{4.3}$$

Solutions satisfying the required boundary conditions (3.8) and (3.9) may be found in textbooks on fluid mechanics such as Lamb (1932) and Milne-Thomson (1967) so giving a set of possible equilibrium solutions for intruded lenses. It is not, however, known *a priori* what the distribution of properties will be *inside* the lenses for these solutions, as they correspond to particular *shapes* in transformed space. What is done, therefore, is to take the well-known potential flow solutions for flow past elliptical obstacles, to reinterpret them as solutions for the intrusion problem and to find out what distributions of properties within the intruded lenses are implied.

As a first example, take the solution for flow past a circular cylinder. This solution has the form

$$w = Y + 1/Y, \tag{4.4}$$

or, taking real and imaginary parts,

$$x = M + M/(M^2 + Z^2), \tag{4.5}$$

$$\theta = Z - Z/(M^2 + Z^2). \tag{4.6}$$

The boundary $\theta = 0$ of the lens is given by

$$M^2 + Z^2 = 1. \tag{4.7}$$

On this boundary, (4.5) shows that

$$x = 2M, \tag{4.8}$$

so the boundary in physical space is the ellipse defined by

$$\tfrac{1}{4}x^2 + z^2 = 1. \tag{4.9}$$

The velocity on this boundary is given by (3.11), that is, by

$$v = M - x = -\tfrac{1}{2}x, \tag{4.10}$$

use being made of (4.8).

The flow inside the lens can now be determined, because the thermal wind equation (2.4) requires that $v$ be independent of $z$ in any homogeneous region, and thus be a function of $x$ only. Also $v$ must be continuous across the boundary of the lens, and hence satisfy (4.10) *throughout* the lens. It follows that the vorticity in the lens is anticyclonic and has uniform magnitude equal to half the Coriolis parameter.

The solution for temperature $\theta$ and velocity $v$ is now completely determined, being given parametrically in terms of $M$ and $Z$ by (4.5), (4.6) and by

$$v = -M/(M^2 + Z^2), \tag{4.11}$$

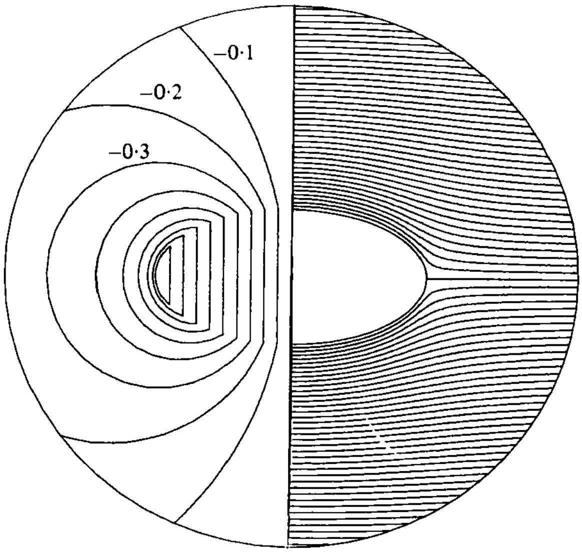


FIGURE 1. Contours of temperature $\theta$ (right side) and velocity $v$ normal to the page (left side) for a homogeneous lens which has the shape of an elliptical cylinder with horizontal semi-axis $2N/f$ times the vertical semi-axis $H$. The diagram is drawn with a vertical exaggeration of $N/f$ where $f$ is the Coriolis parameter and $N$ the buoyancy frequency far from the lens. The velocity $v$ is zero on the vertical axis and increases linearly with distance inside the lens to a maximum value of $NH$ at the extremities. The relative vorticity in the lens thus has a uniform value of $-\frac{1}{2}f$ and is anticyclonic. The total vorticity in the lens is $+\frac{1}{2}f$. In the non-dimensional co-ordinates defined in the text, the contour interval is 0·1 for both $\theta$ and $v$.

which follows from (3.1), for the region outside the lens. Inside the lens, $\theta$ is zero and $v$ is given by (4.10). This solution is shown mapped into $(x, z)$ space in figure 1.

If details of the pressure field are required, these can be obtained by integrating (4.4) since if

$$W = -\phi + \tfrac{1}{2}M^2 + i\psi, \tag{4.12}$$

is the complex quantity such that

$$\frac{dW}{dY} = w, \tag{4.13}$$

it follows that

$$\theta = \frac{\partial}{\partial Z}(\phi - \tfrac{1}{2}M^2) = \frac{\partial \phi}{\partial Z}, \tag{4.14}$$

and that

$$x = -\frac{\partial}{\partial M}(\phi - \tfrac{1}{2}M^2) = M - \frac{\partial \phi}{\partial M}, \tag{4.15}$$

i.e. by (3.11), that

$$v = \frac{\partial \phi}{\partial M}. \tag{4.16}$$

Equations (4.14) and (4.16) are identical with (3.14), showing that $\phi$ is the quantity defined by (3.13). In the present case, (4.4) integrates to give

$$W = \tfrac{1}{2}Y^2 + \ln Y, \tag{4.17}$$

so (4.12) gives

$$\phi = \tfrac{1}{2}Z^2 - \tfrac{1}{2}\ln(M^2 + Z^2). \tag{4.18}$$

It is also useful to define a temperature perturbation $T$ from the original temperature field by

$$T = \theta - Z. \tag{4.19}$$

In the present case, (4.6) shows that $T$ is given by

$$T = -Z/(M^2+Z^2). \tag{4.20}$$

Now that the properties of the solution have been established, it is of interest to consider ways by which such a lens could be generated. Since, in a homogeneous lens, the analogue $M$ of angular momentum is conserved, the volume (per unit length) of fluid with values between $M$ and $M+dM$ is constant. The conservation of this volume may be expressed by the statement that $h$ defined by

$$z\,dx = h\,dM, \tag{4.21}$$

is a function of $M$ only, where $dx$ is the width of fluid in the required range of $M$ and $z$ is half the height of the corresponding column. In the present case (4.8), (4.9) and (4.21) give as the relation between $h$ and $M$

$$(\tfrac{1}{2}h)^2+M^2 = 1. \tag{4.22}$$

Ways of introducing fluid to give this distribution can now be investigated. For instance, the fluid could be introduced through a set of line sources on the plane $z = 0$ such that the introduced fluid has no transverse velocity component $v$ when it emerges. It follows from (3.11) that $M$ is equal to the co-ordinate $x$ of the point at which the fluid emerges. Since the range of $M$ is

$$|M| < 1,$$

it follows that the line sources must occupy this span, and that the volume of fluid emitted at the point with co-ordinate $M$ must be proportional to $h$, i.e. given by (4.22).

The above method is perhaps the simplest way of producing such a lens in the laboratory, but it is also possible, conceptually at least, to imagine other methods. For instance, if the fluid could be inserted into a flexible bag which is made to take the shape given by (4.22) and then brought to rest, this could serve as an initial condition for a Rossby-type adjustment problem. At some initial time, one has to imagine the bag containing the intrusion to dissolve suddenly. Waves would be radiated outward during the adjustment process, but the final state should be the same as found above. The form of potential vorticity appropriate to a homogeneous lens, namely total vorticity divided by depth, is, of course, conserved because

$$\frac{1+dv/dx}{z} = \frac{dM}{z\,dx} = \frac{1}{h}. \tag{4.23}$$

## 5. Solution for a thick lens

A family of solutions with that of the last paragraph as a special case can be obtained by utilizing the solution for irrotational flow past an ellipse (Lamb 1932, § 71; Milne-Thomson 1967, p. 161). The form depends on whether the lens is taller or more squat in transformed space than the lens considered in the last section. Taller lenses will be considered in this section and will be called *thick lenses*. These lenses turn out to be elliptical in a physical space as well as transformed space, and 'thick' lenses are ones with aspect ratios (vertical axis over horizontal axis) greater than one half.

For describing the solution in these cases, elliptic co-ordinates

$$\zeta = \xi + i\eta \tag{5.1}$$

are defined by the transformation

$$Y = \operatorname{sech} \xi_0 \sinh \zeta \tag{5.2}$$

that is, by

$$M = \operatorname{sech} \zeta_0 \sinh \xi \cos \eta, \tag{5.3}$$

$$Z = \operatorname{sech} \xi_0 \cosh \xi \sin \eta. \tag{5.4}$$

The boundary of the lens is defined by $\xi = \xi_0$, and on this boundary

$$M = \tanh \xi_0 \cos \eta, \quad Z = \sin \eta \quad \text{on} \quad \xi = \xi_0. \tag{5.5}$$

The solution for flow past this ellipse (Lamb § 71, Milne-Thomson p. 164) is

$$w = a \cosh (\zeta - \xi_0) = Y + e^{\xi_0 - \zeta}, \tag{5.6}$$

where

$$a = 1 + \tanh \xi_0 = e^{\xi_0} \operatorname{sech} \xi_0. \tag{5.7}$$

In terms of real and imaginary parts, this solution is

$$x = a \cosh (\xi - \xi_0) \cos \eta = M + e^{\xi_0 - \xi} \cos \eta, \tag{5.8}$$

$$\theta = a \sinh (\xi - \xi_0) \sin \eta = Z - e^{\xi_0 - \xi} \sin \eta \tag{5.9}$$

and so by (3.11) and (4.20) the velocity $v$ and temperature $T$ are given by

$$V = -e^{\xi_0 - \xi} \cos \eta, \tag{5.10}$$

$$T = -e^{\xi_0 - \xi} \sin \eta. \tag{5.11}$$

On the lens boundary $\xi = \xi_0$, (5.8), (5.9) and (5.10) give

$$x = a \cos \eta, \quad \theta = 0, \quad v = -\cos \eta \quad \text{on} \quad \xi = \xi_0. \tag{5.12}$$

It follows that the lens in physical space is elliptical with half-width $a$, i.e. in dimensional terms

$$\text{half-width of elliptical lens} = aNH/f. \tag{5.13}$$

It also follows that on the edge of the lens

$$v = -x/a \tag{5.14}$$

and so this gives the velocity distribution inside the lens. The *vorticity* is uniform inside the lens, and anticyclonic with magnitude $1/a$ times the Coriolis parameter.

The function $h(M)$ can be evaluated for these lenses by the same methods as in the last section. A case of special significance for laboratory experiments is the one where $a = 1$, for then (3.11) and (5.14) give

$$M = x + v = 0, \tag{5.15}$$

i.e. $M$ is uniformly zero inside the lens. This corresponds to the case where the intrusion is made by introducing fluid through a *narrow* slit at $x = 0$ (cf. Gill *et al.* (1979), so only a single line source is needed. Griffiths & Linden (1981) in fact report an experiment

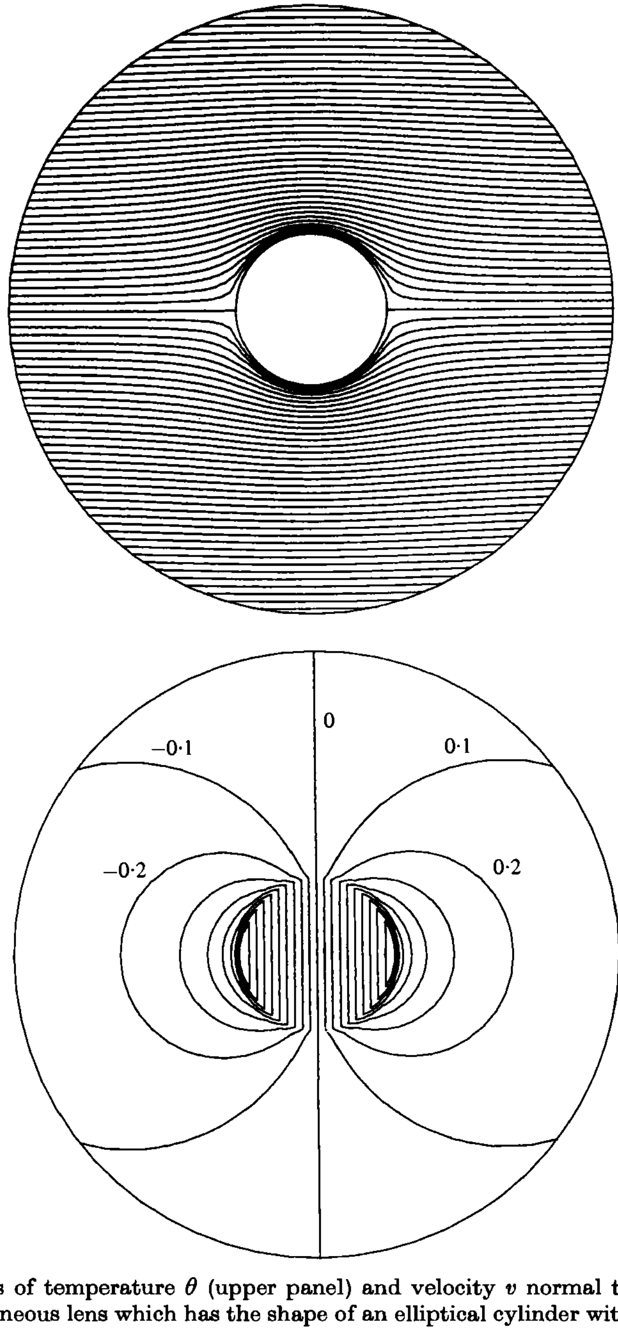


FIGURE 2. Contours of temperature $\theta$ (upper panel) and velocity $v$ normal to the page (lower panel) for a homogeneous lens which has the shape of an elliptical cylinder with horizontal semi-axis $N/f$ times the vertical semi-axis $H$. The diagram is drawn with a vertical exaggeration of $f/N$. The velocity $v$ is zero on the vertical axis and the relative vorticity inside the lens has the uniform value $-f$. Thus the total vorticity is zero inside the lens, and the maximum value of $v$ is $NH$. In the non-dimensional co-ordinates defined in the text, the contour interval is 0·1 for both $\theta$ and $v$.

which is the axisymmetric analogue of this case. The corresponding value of $\xi_0$ by (5.7) is zero. Thus the solution outside the lens is given by

$$\left.\begin{aligned} x &= \cosh\xi\cos\eta, & Z &= \cosh\xi\sin\eta, \\ \theta &= \sinh\xi\sin\eta, & v &= -e^{-\xi}\cos\eta. \end{aligned}\right\} \tag{5.16}$$

The boundary of the lens is circular and, if $r$, $\eta$ are polar co-ordinates, the solution can also be written

$$\theta = \left\{\begin{matrix} (r^2-1)^{\frac{1}{2}}\sin\eta & \text{for} & r \geqslant 1, \\ 0 & \text{for} & r \leqslant 1, \end{matrix}\right\} \tag{5.17}$$

and

$$v = \left\{\begin{matrix} -(r-(r^2-1)^{\frac{1}{2}})\cos\eta & \text{for} & r \geqslant 1, \\ -r\cos\eta & \text{for} & r \leqslant 1. \end{matrix}\right\} \tag{5.18}$$

This solution is shown in figure 2.

## 6. Solution for a thin lens

When the lens is thinner than the one considered in § 4, the appropriate elliptic co-ordinates may be defined by

$$Y = \operatorname{cosech}\xi_0\cosh\zeta, \tag{6.1}$$

that is by

$$M = \operatorname{cosech}\xi_0\cosh\xi\cos\eta, \tag{6.2}$$

$$Z = \operatorname{cosech}\xi_0\sinh\xi\sin\eta. \tag{6.3}$$

The boundary of the ellipse is defined by $\xi = \xi_0$ and on this boundary

$$M = \coth\xi_0\cos\eta, \quad Z = \sin\eta \quad \text{on} \quad \xi = \xi_0. \tag{6.4}$$

The solution is again given by (5.6), but the relationship between $a$ and $\xi_0$ is different, namely

$$a = 1+\coth\xi_0 = e^{\xi_0}\operatorname{cosech}\xi_0. \tag{6.5}$$

Otherwise, the solutions have the same form (5.8), (5.9), (5.10) and (5.11) as before. (5.12) still applies on the lens boundary and (5.14) still applied within the lens. The half-width of the lens is still given by (5.13). The difference is that in the last section $a$ was restricted to the range $1 \leqslant a \leqslant 2$ whereas in this section it occupies the range $a \geqslant 2$. The borderline case $a = 2$ is the one considered in § 4. Figure 3 shows the solution for the case $a = 4$.

An especially interesting limit is the one where $a \to \infty$ or $\xi_0 \to 0$, i.e. the limit where the lens becomes very thin. At large distances from the origin ($\xi \gg 1$, i.e. radial distance $r = (x^2+z^2)^{\frac{1}{2}} \gg a$), the shape of the lens does not matter and the disturbance properties are determined by the major semi-axis $a$. The approximate solution for the disturbance, as for all cases considered in this paper, is given by

$$v = -\frac{a}{r}\cos\eta, \quad T = -\frac{a}{r}\sin\eta, \tag{6.6}$$

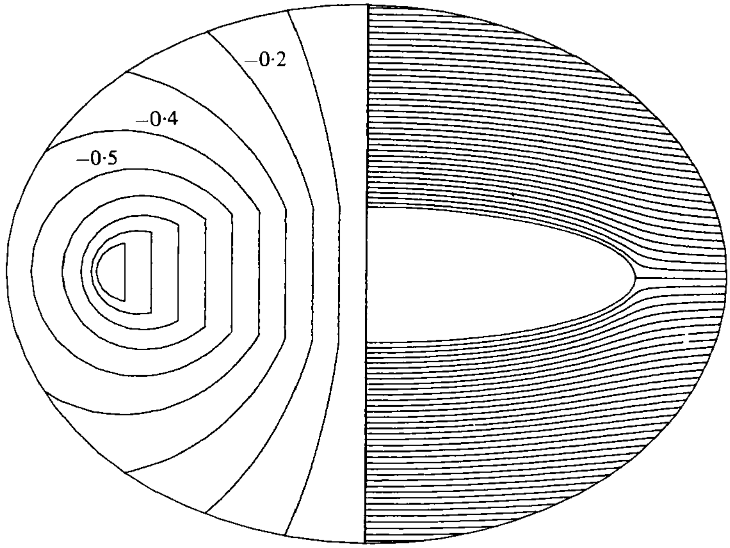


FIGURE 3. As for figure 1, except that the lens has aspect ratio $4N/f$. The maximum velocity is again $NH$ so the relative vorticity inside the lens is $-\frac{1}{4}f$.

where $(r,\eta)$ are polar co-ordinates. For $\xi \ll 1$, on the other hand, the approximation including terms of order $\xi$ gives

$$\left.\begin{aligned} T &= -(1-x^2/a^2)^{\frac{1}{2}} + \frac{1}{a}(z-(1-x^2/a^2)^{\frac{1}{2}}), \\ v &= -\frac{x}{a}\left(1+\frac{1}{a}(1-z(1-x^2/a^2)^{-\frac{1}{2}})\right), \end{aligned}\right\} \tag{6.7}$$

this being valid outside the lens boundary, i.e. for

$$z^2 > 1-x^2/a^2.$$

The solution is approximately linear, as can be seen from writing the non-dimensional form of (2.5) in terms of $v$ and $T$, $T$ being defined by (4.19). The result is

$$q-1 = \left(\frac{\partial v}{\partial x}+\frac{\partial T}{\partial z}\right)+\left(\frac{\partial v}{\partial x}\frac{\partial T}{\partial z}-\frac{\partial v}{\partial z}\frac{\partial T}{\partial x}\right), \tag{6.8}$$

and in the present case the nonlinear term is of order $1/a^2$ and so small compared with the linear term which is of order $1/a$. This result is not uniformly true, however, because the approximation (6.7) breaks down near the 'nose' $x = a$, $z = 0$ of the intrusion where the transformation from $(x,z)$ to $(M,Z)$ space is singular. This singularity will now be examined in some detail.

## 7. Solution near the nose of a very thin lens

The transformation between the $(M,Z)$ and $(\xi,\eta)$ planes is a conformal transformation and therefore regular. In particular

$$\partial M/\partial\xi = \operatorname{cosech}\xi_0 \sinh\xi\cos\eta, \quad \partial M/\partial y = -\operatorname{cosech}\xi_0\cosh\xi\sin\eta, \tag{7.1}$$

$$\partial Z/\partial\xi = \operatorname{cosech}\xi_0\cosh\xi\sin\eta, \quad \partial Z/\partial\eta = \operatorname{cosech}\xi_0\sinh\xi\cos\eta \tag{7.2}$$

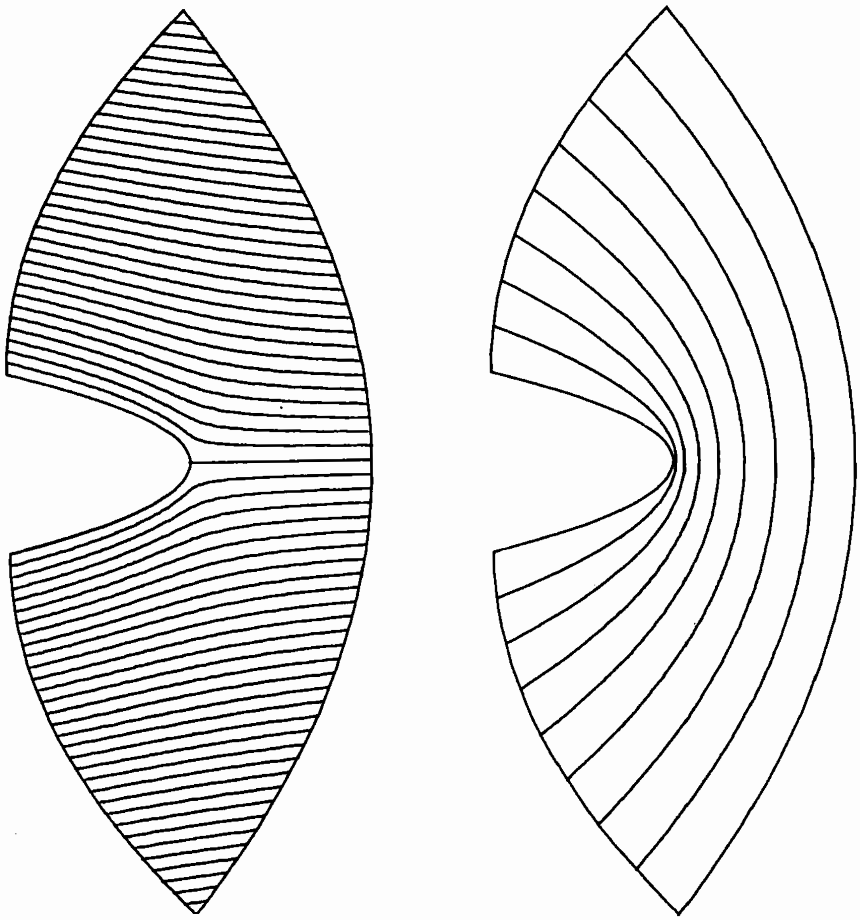

FIGURE 4. Contours of temperature $\theta$ (left panel) and velocity $v$ normal to the page (right panel) near the 'nose' of a very thin intrusive lens. The region shown is defined by $0 \leqslant \xi' \leqslant 4$, $|\eta'| \leqslant 4$ and the contour interval for the non-dimensional variables $\theta'$ and $v'$ is 0·5.

and the Jacobian of the transformation is

$$\frac{\partial(M,Z)}{\partial(\xi,\eta)} = \operatorname{cosech}^2 \xi_0 (\sinh^2 \xi \cos^2 \eta + \cosh^2 \xi \sin^2 \eta). \tag{7.3}$$

However, the $x$-derivatives are, by (5.8),

$$\partial x/\partial \xi = a \sinh(\xi - \xi_0) \cos \eta, \quad \partial x/\partial \eta = -a \cosh(\xi - \xi_0) \sin \eta, \tag{7.4}$$

and *both* derivatives vanish at $\xi = \xi_0$, $\eta = 0$ so the Jacobian $\partial(x,z)/\partial(\xi,\eta)$ vanishes at this point and the transformation is irregular. At the same point, the Jacobian given by (7.3) is equal to unity.

The solution in the neighbourhood of the singularity can be obtained by expanding variables as follows:

$$\zeta - \xi_0 = \xi_0 \zeta', \tag{7.5}$$

$$Y - \coth \xi_0 = \xi_0 Y', \tag{7.6}$$

$$w - a = \xi_0 w'. \tag{7.7}$$

Then, to a first approximation, (6.1) and (5.6) become

$$Y' = \zeta' + \tfrac{1}{2}\zeta'^2, \quad w' = \tfrac{1}{2}\zeta'^2, \tag{7.8), (7.9}$$

and these define the solution in the neighbourhood of the nose. (7.6) and (7.7) show that this neighbourhood has dimensions of order $\xi_0$ or $1/a$ in physical space.

Using the prime to denote scaled perturbations from values at the nose for all variables (7.8) and (7.9) can be written in terms of real and imaginary parts as

$$M' = \xi' + \tfrac{1}{2}(\xi'^2 - \eta'^2), \tag{7.10}$$

$$Z' = \eta'(1+\xi'), \tag{7.11}$$

$$x' = \tfrac{1}{2}(\xi'^2 - \eta'^2), \tag{7.12}$$

$$\theta' = \xi'\eta'. \tag{7.13}$$

In addition, the perturbation forms of (3.11) and (4.19) give

$$v' = \xi', \quad T' = -\eta'. \tag{7.14), (7.15}$$

The solution is shown in figure 4.

## 8. Layered structure observed in experiments

Griffiths & Linden (1981) report an experiment which corresponds to the axisymmetric equivalent of the lens with $M = 0$ described in § 4. An interesting feature of that experiment was the development of a layered structure similar in character to that which Baker (1971) attributed to a diffusive mechanism (McIntyre 1970) by which available potential energy in a rotating stratified fluid can be released. The condition given by McIntyre (1970, e.g. (3.3(*a*)) for instability to occur is, in the present notation,

$$\frac{\partial M}{\partial x}\frac{\partial \theta}{\partial z}\bigg/\left(\frac{\partial v}{\partial z}\right)^2 < \frac{(\sigma+1)^2}{4\sigma}, \tag{8.1}$$

where

$$\sigma = \nu/\kappa, \tag{8.2}$$

is the Prandtl number of the fluid. Using the non-dimensional forms of (2.4) and (2.5) with $q$ given by (2.6), this reduces to the condition that the shear $S = \partial v/\partial z$ has to satisfy

$$S^2 > \frac{4\sigma}{(\sigma-1)^2}, \tag{8.3}$$

for instability. For example, when the Prandtl number is 10, $S$ must be greater than 0·7 for instability to occur. At best, the condition (8.3) can only be regarded as a guide to where instability might occur because (*a*) the solutions are obtained from inviscid theory, and (*b*) the criterion derived by McIntyre was for a field with uniform $S$. However, it seems worth while to calculate values of $S$ for the solutions of the previous sections and this can be done using Jacobians, e.g.

$$S = \frac{\partial(x,v)}{\partial(x,z)} = \frac{\partial(x,v)}{\partial(\xi,\eta)}\bigg/\frac{\partial(x,z)}{\partial(\xi,\eta)}. \tag{8.4}$$

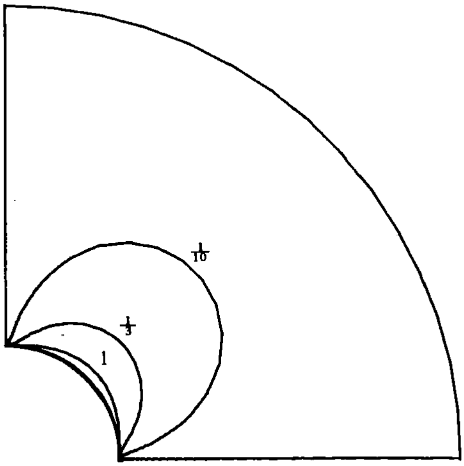


FIGURE 5. Contours of shear $S = \partial v/\partial z$ for the case depicted in figure 2, namely an elliptic cylinder with horizontal semi-axis $N/f$ times the vertical semi-axis. Values are given in non-dimensional units using the shear scale $N$. For fluid of Prandtl number 10, a diffusive instability can occur when $S$ is greater than 0·7.

For instance, for the lens which has zero absolute vorticity (the case which seems most likely to be relevant to laboratory experiment), substituting from (5.16) in (8.4) gives

$$S = \frac{\sin 2\eta}{\sinh 2\xi} = \frac{\sin\eta\cos\eta}{r(r^2-1)^{\frac{1}{2}}}. \tag{8.5}$$

It is immediately apparent that the shear is *infinite* on the lens boundary $r = 1$, thus demonstrating the singular character of the inviscid solution. Contours of $S$ are shown in figure 5. At any radius, $S$ is maximum at $\eta = \frac{1}{4}\pi$ so one might expect some preference for instability to occur near the lens boundary at this location.

## 9. Spin-down of an intrusion through viscous effects

Although the solutions described above are calculated for an inviscid fluid, one might gain some idea about the effect of friction by calculating the viscous shear forces that would be produced in the above solutions if viscosity were suddenly 'switched on' and the motion resulting from these forces calculated. When this is done, it is found that a flow is produced which will tend to flatten out the lens. However, the strongest effects are located near the singular features of the inviscid solution, namely the discontinuity in temperature and velocity gradients at the edge of the lens and the singularity at the nose. At this point, the effects are particularly strong, the equations are strongly nonlinear and the expected result would be for the singularity causing the trouble to be removed in a very short time.

In practice, dissipative effects would be operative during the formation of the lens and those would oppose the formation of singular features. One might expect, therefore, to produce a temperature distribution approximately the same as the inviscid one *except* in the neighbourhood of the lens boundary, where the discontinuities in gradient would be smoothed out. Supposing such a state is attained, and no further fluid is intruded into the system, it is of interest to calculate how the lens 'spins down' towards a state of rest under the action of viscous forces. It turns out that this process has a very simple character for very thin lenses, as then the motion is quasi-geostrophic. The process itself elongates the lenses, so causing them to better approximate a quasi-geostrophic balance as time goes on, and hence is likely to describe the asymptotic state of any intrusive feature.

The motion in the above solution is in the $y$-direction and so, in real fluids, there will be viscous stresses acting to accelerate the fluid in this direction. Since the scale ratio $N/f$ (the ratio of horizontal to vertical scales) is usually large in geophysical applications, the viscous driving term in the $y$-momentum equation can be approximated by $\nu\, \partial^2 v/\partial z^2$. If the adjustment is slow and the relative vorticity is small compared with $f$ (as it is for very thin lenses), the balancing term is the Coriolis term associated with flow in the $x$ direction, i.e. the approximate equation is, in dimensional form,

$$fu = \nu \frac{\partial^2 v}{\partial z^2}. \tag{9.1}$$

Also, since there is no dependence on $y$, the continuity equation allows the introduction of a stream function $\psi$ defined by

$$u = \frac{\partial \psi}{\partial z}, \quad w = -\frac{\partial \psi}{\partial x}. \tag{9.2}$$

Substituting in (9.1) and integrating with respect to $z$, the result is

$$f\psi = \nu \frac{\partial v}{\partial z} = \frac{\alpha g \nu}{f} \frac{\partial \theta}{\partial x}, \tag{9.3}$$

i.e. the stream function is proportional to the shear (the quantity discussed in § 8) and this is proportional to the horizontal temperature gradient by the thermal-wind equation (2.4). If $H$ is an appropriate measure of the vertical scale, it can be shown that the geostrophic balance associated with the $x$ component of the momentum equation and the hydrostatic balance associated with the $z$ component are not upset by viscous effects, provided the Ekman number $E$ defined by

$$E = \frac{\nu}{fH^2} \tag{9.4}$$

is small.

It remains to consider the temperature equation

$$\frac{\partial \theta}{\partial t} + \frac{\partial \psi}{\partial z}\frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial z} = \kappa\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial z^2}\right),$$

i.e., substituting from (9.3),

$$\frac{\partial \theta}{\partial t} + \frac{\alpha g \nu}{f^2}\left(\frac{\partial^2 \theta}{\partial x \partial z}\frac{\partial \theta}{\partial x} - \frac{\partial^2 \theta}{\partial x^2}\frac{\partial \theta}{\partial z}\right) = \kappa\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial z^2}\right). \tag{9.5}$$

But for very thin lenses (see § 6) the temperature field is a small perturbation from the initial state of uniform stratification given by (2.1). Applying this to (9.5) simplifies the Jacobian term, and reduces it to

$$\frac{\partial \theta}{\partial t} = \left(\kappa + \frac{N^2 \nu}{f^2}\right)\frac{\partial^2 \theta}{\partial x^2} + \kappa \frac{\partial^2 \theta}{\partial z^2}. \tag{9.6}$$

In other words, the spin-down process produces flow in the $x$, $z$ plane which spreads out the intrusion in the same way as a lateral diffusion process would, the corresponding lateral diffusivity being

$$\frac{N^2}{f^2}\nu. \tag{9.7}$$

This is reminiscent of the effect of spin-down on the interface of a two-layer system, as this also acts as a lateral diffusion (Gill *et al.* 1979).

The spin-down lateral diffusivity $\nu N^2/f^2$ is much greater than the natural diffusivity $\kappa$ because $N/f$ is large and also because the Prandtl number

$$\sigma = \nu/\kappa \tag{9.8}$$

is large. For typical values in the ocean, for instance, the spin-down diffusivity would be about 10000 times $\kappa$. Thus the term involving $\kappa$ in the lateral diffusion term of (9.6) can be ignored. The equation can now be put in non-dimensional form by using the scales defined in § 3 with the addition of a time scale

$$H^2/\nu. \tag{9.9}$$

Then (9.6) becomes simply

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2} + \frac{1}{\sigma}\frac{\partial^2 \theta}{\partial z^2}. \tag{9.10}$$

It should be remembered that, although the time scale (9.9) is associated with a diffusion-like term in (9.6), it is better interpreted as the time required to flatten an isotherm by advection at a speed of the order given by (9.1). The value of (9.9) in a laboratory situation with half-height $H = 5$ cm is about 40 min, whereas, for an oceanographic example with $H = 10$ m, the time to spin down by molecular processes with $\nu = 10^{-6}\,\mathrm{m^2\,s^{-1}}$ would, according to (9.9), be about three years. This estimate, however, assumes the horizontal scale is $NH/f$. For a larger horizontal scale $L$, (9.6) gives two possible decay times

$$\frac{f^2L^2}{N^2\nu} \quad \text{and} \quad \frac{H^2}{\kappa}, \tag{9.11}$$

which are both larger than (9.9).

To illustrate the spin-down effect, equation (9.10) was solved numerically for the initial state illustrated in figure 3. The assumptions on which the theory is based are not satisfied initially because of the singularities, but these are soon removed by the diffusion process. Figure 6 shows the result.

The main effect showing at time $t = 1$ in the temperature field is in the region of the lens. In particular, the isotherm $\theta = 0{\cdot}2$ has moved downward in this region under the action of the viscosity-induced motion by distances between 0·3 and 0·4 units. This represents a substantial amount of flattening. The velocity field which is responsible for the spin-down is shown in figure 6($c$) at $t = 1$. As the horizontal temperature gradient reduces, the velocity $v$ normal to the $(x, z)$ phase also reduces in accordance with the thermal-wind equation. Figure 6($b$) shows this effect which is greatest near the lateral extremities of the intrusion.

With time, the intrusion would continue to spread and weaken. In the process, one would expect the ratio of horizontal and vertical scales to adjust so that the two decay times (9.11) become roughly equal. Otherwise, the process with the shorter time scale would dominate and increase the associated scale. This would increase the

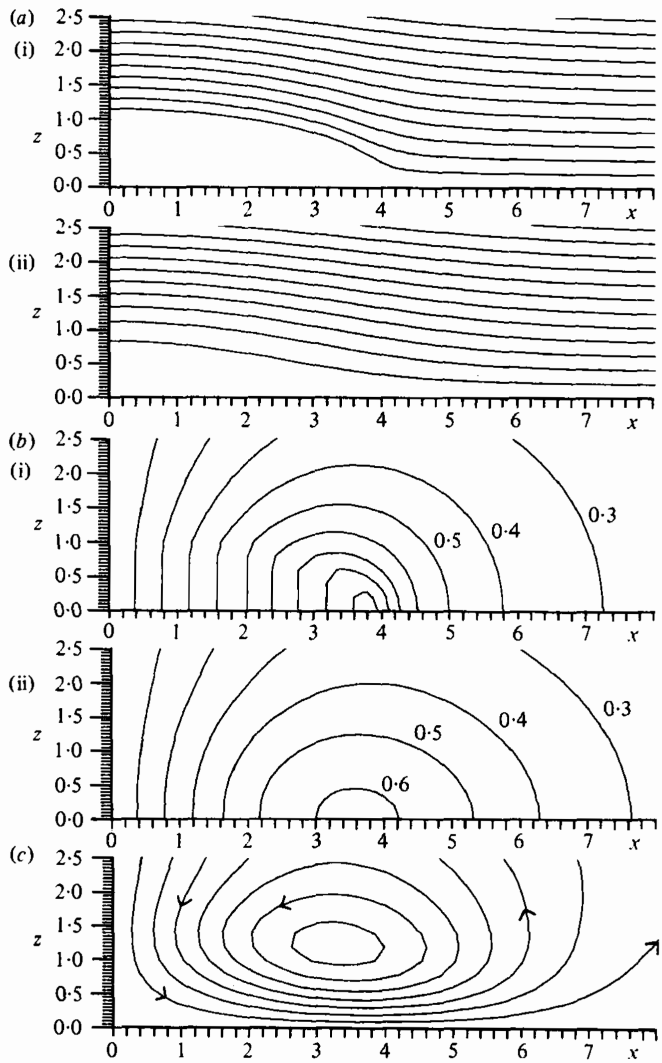


FIGURE 6. (a) Contours of temperature $\theta$ in the positive quarter-plane at (i) $t = 0$ and (ii) $t = 1$. The initial field is the same as shown in figure 3 while the second panel shows the result of spin-down as calculated from equation (9.10). The contour interval is 0·2. (b) The velocity $v$ normal to the page at (i) $t = 0$ and (ii) $t = 1$. The contour interval is 0·1. (c) The stream function for flow in the $(x, z)$ plane at $t = 1$. The contour interval is 0·02 using a stream function scale of $\nu N/f$.

corresponding time scale until the two matched. Thus the ultimate scale ratio would be expected to satisfy

$$\frac{L}{H} \sim \frac{\sigma^{\frac{1}{2}} N}{f}. \tag{9.12}$$

## 10. Axisymmetric lenses

The foregoing theory has been entirely concerned with two-dimensional configurations where there is no dependence on $y$. In laboratory experiments, the arrangements usually give rise to axial symmetry, so it is worth investigating how far the above techniques can be carried in this case. The theory, it turns out, is not as straightforward because the Cauchy–Riemann equations are not obtained. If $r$ is radial distance from the axis and $v$ is now azimuthal velocity, the thermal wind equation in non-dimensional form (scales as before) is

$$\frac{\partial v}{\partial z}\frac{r+2v}{r} = \frac{\partial \theta}{\partial r}, \tag{10.1}$$

while the expression for Ertel's potential vorticity $q$ is

$$q = \left(1+\frac{\partial v}{\partial r}+\frac{v}{r}\right)\frac{\partial \theta}{\partial z} - \frac{\partial v}{\partial z}\frac{\partial \theta}{\partial r}. \tag{10.2}$$

As before, it is helpful to introduce a new variable in place of $v$ which will again be denoted by $M$, but now $M$ signifies the angular momentum about the axis of symmetry, that is

$$M = rv + \tfrac{1}{2}r^2. \tag{10.3}$$

It is useful to replace the radial co-ordinate $r$ with

$$R = \tfrac{1}{2}r^2. \tag{10.4}$$

Now the potential vorticity equation (10.2) with the specified potential vorticity $q = 1$ takes on the Jacobian form

$$\frac{\partial(M,\theta)}{\partial(R,z)} = 1, \tag{10.5}$$

but the thermal wind equation is not as simple as before. It now has the form

$$\frac{\partial M}{\partial z}\frac{M}{2R^2} = \frac{\partial \theta}{\partial R} \quad \text{or} \quad \frac{\partial(\theta,z)}{\partial(M,R)} = -\frac{2R^2}{M}. \tag{10.6}$$

If $M$ and $Z$ are chosen as independent variables (capital $Z$ being used for partial derivatives keeping $M$ constant), the two equations are

$$\frac{\partial \theta}{\partial Z} = \frac{\partial R}{\partial M} \quad \text{and} \quad \frac{1}{2R^2}\frac{\partial R}{\partial Z} = -\frac{1}{M}\frac{\partial \theta}{\partial M}. \tag{10.7}$$

These are still nonlinear equations, so there is no obvious way of finding exact solutions apart from guessing particular forms. However, for lenses which have the property

$$2v \ll r \tag{10.8}$$

everywhere, the factor $M/2R^2$ in (10.7) is approximately equal to $1/2M$ and the second of (10.7) becomes

$$\frac{\partial R}{\partial Z} = -2M\frac{\partial \theta}{\partial M}. \tag{10.9}$$

Eliminating $R$ and introducing the variable $s$ defined by (cf. (10.4))

$$M = \tfrac{1}{2}s^2, \tag{10.10}$$

the result is Laplace's equation

$$\frac{\partial^2 \theta}{\partial Z^2} + \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial \theta}{\partial s}\right) = 0 \tag{10.11}$$

as demonstrated in a more general context by Hoskins (1975). Thus it is possible to find approximate solutions for very thin lenses. For such lenses, spin-down effects can again be calculated using the procedure of § 9. The result is the same, namely that the viscosity-induced motion has the same effect on the temperature field as an enhanced lateral diffusivity with magnitude given by (9.7).

## 11. Discussion

Apart from their appeal as exact solutions for fluid flow, the results found for inviscid intrusions do give some useful properties. In particular, for the 'laboratory case' ($M = 0$ inside the lens), the theory indicates that the ratio of the horizontal dimension to the vertical dimension of the lens should be about $N/f$ and experiments by Rudels (private communication, Woods Hole GFD Program 1979) and by Griffiths & Linden (1981) confirm this. The maximum 'swirl' velocity should be about $NH \sim fL$, where $H$ is the half-depth of the lens and $L$ the half-width and the decay distance over which effects of the intrusion are felt is of order $H$ in the vertical and of order $L$ in the horizontal.

The discussion of viscous effects leads to the conclusion that spin-down acts on the temperature field like an enlarged lateral diffusion. Hence the intrusion becomes larger and weaker with time due to a combination of spin-down and vertical diffusion. The ultimate value for the ratio of horizontal to vertical dimension of the lens is of order $\sigma^{\frac{1}{2}}N/f$, where $\sigma$ is the Prandtl number. For water, the value of $\sigma^{\frac{1}{2}}$ is only 3, so the aspect ratio of a lens 'run down' by molecular processes would not exceed $N/f$ by a very large factor.

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